Problem 2.7

Find the electric field a distance z from the center of a spherical surface of radius R (Fig. 2.11) that carries a uniform charge density σ. Treat the case $z < R$ (inside) as well as $z > R$ (outside). Express your answers in terms of the total charge q on the sphere. [*Hint:* Use the law of cosines to write $\mathscr{r}$ in terms of R and θ. Be sure to take the *positive* square root: $\sqrt{R^2 + z^2 - 2Rz} = (R - z)$ if $R > z$, but it's $(z - R)$ if $R < z$.]

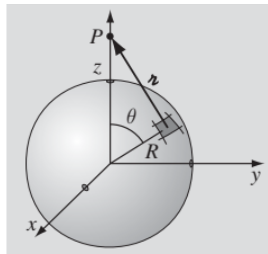
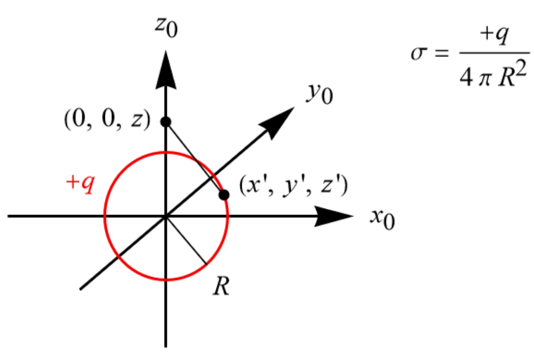


Fig. 2.11

Solution

Start by drawing a schematic for some point on the sphere.



The formula for the electric field from a continuous distribution of charge on a surface is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,da' = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right)da'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,da',$$

where the integral is taken over the surface where the charge exists. Note that $\mathbf{r}$ is the position vector to where we want to know the electric field, $\mathbf{r}'$ is the position vector to the point we chose on the surface, and $\mathscr{r} = |\mathbf{r}-\mathbf{r}'|$ is the distance from the point we chose on the surface to where we want to know the electric field.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\iint\limits_{x_0^2+y_0^2+z_0^2=R^2} \frac{\sigma}{\left[\sqrt{(0-x')^2+(0-y')^2+(z-z')^2}\right]^3}(\langle 0,0,z\rangle - \langle x',y',z'\rangle)\,dS'$$

The surface is spherical, so the appropriate parameterization is done with spherical coordinates (r_0, ϕ_0, θ_0), where θ_0 is the angle from the polar axis.

$$\mathbf{r}' = R\langle \cos\phi' \sin\theta', \sin\phi' \sin\theta', \cos\theta' \rangle, \quad 0 \le \phi' \le 2\pi, \ 0 \le \theta' \le \pi$$

Consequently, the electric field at $\mathbf{r} = \langle 0, 0, z \rangle$ is

$$\mathbf{E} = \frac{\sigma}{4\pi\epsilon_0} \int_0^\pi \int_0^{2\pi} \frac{1}{\left[\sqrt{(0 - R\cos\phi' \sin\theta')^2 + (0 - R\sin\phi' \sin\theta')^2 + (z - R\cos\theta')^2}\right]^3} \left(\langle 0, 0, z\rangle - R\langle \cos\phi' \sin\theta', \sin\phi' \sin\theta', \cos\theta' \rangle\right) \left(R^2 \sin\theta' \, d\phi' \, d\theta'\right)$$

$$= \frac{\sigma R^2}{4\pi\epsilon_0} \int_0^\pi \int_0^{2\pi} \frac{1}{\left[R^2 \sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}} \langle -R\cos\phi' \sin\theta', -R\sin\phi' \sin\theta', z - R\cos\theta' \rangle \left(\sin\theta' \, d\phi' \, d\theta'\right)$$

$$= \frac{\sigma R^2}{4\pi\epsilon_0} \left\langle -R \int_0^\pi \int_0^{2\pi} \frac{\cos\phi' \sin^2\theta'}{\left[R^2 \sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}} \, d\phi' \, d\theta', -R \int_0^\pi \int_0^{2\pi} \frac{\sin\phi' \sin^2\theta'}{\left[R^2 \sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}} \, d\phi' \, d\theta', \right.$$

$$\left. \int_0^\pi \int_0^{2\pi} \frac{(z - R\cos\theta') \sin\theta'}{\left[R^2 \sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}} \, d\phi' \, d\theta' \right\rangle$$

$$= \frac{\sigma R^2}{4\pi\epsilon_0} \left\langle -R \left(\int_0^{2\pi} \cos\phi' \, d\phi'\right) \left\{\int_0^\pi \frac{\sin^2\theta'}{\left[R^2 \sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}} \, d\theta'\right\}, \right.$$

$$- R \left(\int_0^{2\pi} \sin\phi' \, d\phi'\right) \left\{\int_0^\pi \frac{\sin^2\theta'}{\left[R^2 \sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}} \, d\theta'\right\},$$

$$\left. \left(\int_0^{2\pi} d\phi'\right) \left\{\int_0^\pi \frac{(z - R\cos\theta') \sin\theta'}{\left[R^2 \sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}} \, d\theta'\right\} \right\rangle$$

$$= \frac{\sigma R^2}{4\pi\epsilon_0} \left\langle -R(0) \left\{\int_0^\pi \frac{\sin^2\theta'}{\left[R^2 \sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}} \, d\theta'\right\}, -R(0) \left\{\int_0^\pi \frac{\sin^2\theta'}{\left[R^2 \sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}} \, d\theta'\right\}, \right.$$

$$\left. (2\pi) \left\{\int_0^\pi \frac{(z - R\cos\theta') \sin\theta'}{\left[R^2 \sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}} \, d\theta'\right\} \right\rangle.$$

Continue the simplification.

$$\begin{aligned}
\mathbf{E} &= \frac{\sigma R^2}{4\pi\epsilon_0}\left\langle 0, 0, 2\pi \int_0^\pi \frac{(z - R\cos\theta')\sin\theta'}{\left[R^2\sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}}\, d\theta' \right\rangle \\
&= \frac{\sigma R^2}{2\epsilon_0}\langle 0, 0, 1\rangle \int_0^\pi \frac{(z - R\cos\theta')\sin\theta'}{\left[R^2\sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}}\, d\theta' \\
&= \frac{\sigma R^2\hat{\mathbf{z}}}{2\epsilon_0}\int_0^\pi \frac{(z - R\cos\theta')\sin\theta'}{\left[R^2\sin^2\theta' + (z - R\cos\theta')^2\right]^{3/2}}\, d\theta' \\
&= \frac{\sigma R^2\hat{\mathbf{z}}}{2\epsilon_0}\int_0^\pi \frac{(z - R\cos\theta')\sin\theta'}{[R^2(1 - \cos^2\theta') + (z - R\cos\theta')^2]^{3/2}}\, d\theta' \\
&= \frac{\sigma R^2\hat{\mathbf{z}}}{2\epsilon_0}\int_0^\pi \frac{(z - R\cos\theta')\sin\theta'}{[R^2 - R^2\cos^2\theta' + (z - R\cos\theta')^2]^{3/2}}\, d\theta'
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
u &= z - R\cos\theta' &\quad\rightarrow\quad& R\cos\theta' = z - u \\
du &= R\sin\theta'\, d\theta' &\quad\rightarrow\quad& \frac{du}{R} = \sin\theta'\, d\theta'
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathbf{E} &= \frac{\sigma R^2\hat{\mathbf{z}}}{2\epsilon_0}\int_{z-R\cos 0}^{z-R\cos\pi} \frac{u}{[R^2 - (z-u)^2 + u^2]^{3/2}}\left(\frac{du}{R}\right) \\
&= \frac{\sigma R\hat{\mathbf{z}}}{2\epsilon_0}\int_{z-R}^{z+R} \frac{u}{(2zu + R^2 - z^2)^{3/2}}\, du.
\end{aligned}$$

Make a second substitution.

$$\begin{aligned}
v &= 2zu + R^2 - z^2 &\quad\rightarrow\quad& \frac{v + z^2 - R^2}{2z} = u \\
dv &= 2z\, du &\quad\rightarrow\quad& \frac{dv}{2z} = du
\end{aligned}$$

So then

$$\begin{aligned}
\mathbf{E} &= \frac{\sigma R\hat{\mathbf{z}}}{2\epsilon_0}\int_{2z(z-R)+R^2-z^2}^{2z(z+R)+R^2-z^2} \frac{v + z^2 - R^2}{2zv^{3/2}}\left(\frac{dv}{2z}\right) \\
&= \frac{\sigma R\hat{\mathbf{z}}}{8\epsilon_0 z^2}\int_{z^2-2Rz+R^2}^{z^2+2Rz+R^2} \frac{v + z^2 - R^2}{v^{3/2}}\, dv \\
&= \frac{\sigma R\hat{\mathbf{z}}}{8\epsilon_0 z^2}\int_{z^2-2Rz+R^2}^{z^2+2Rz+R^2} [v^{-1/2} + (z^2 - R^2)v^{-3/2}]\, dv.
\end{aligned}$$

Evaluate the integral and simplify the result.

$$\begin{aligned}
\mathbf{E} &= \frac{\sigma R\hat{\mathbf{z}}}{8\epsilon_0 z^2}\left[(2v^{1/2})\Big|_{z^2-2Rz+R^2}^{z^2+2Rz+R^2} + (z^2-R^2)(-2v^{-1/2})\Big|_{z^2-2Rz+R^2}^{z^2+2Rz+R^2}\right] \\
&= \frac{\sigma R\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left[\left(\sqrt{z^2+2Rz+R^2}-\sqrt{z^2-2Rz+R^2}\right)-(z^2-R^2)\left(\frac{1}{\sqrt{z^2+2Rz+R^2}}-\frac{1}{\sqrt{z^2-2Rz+R^2}}\right)\right] \\
&= \frac{\sigma R\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left[\left(\sqrt{(z+R)^2}-\sqrt{(z-R)^2}\right)-(z^2-R^2)\left(\frac{1}{\sqrt{(z+R)^2}}-\frac{1}{\sqrt{(z-R)^2}}\right)\right] \\
&= \frac{\sigma R\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left[(z+R)-|z-R|-(z^2-R^2)\left(\frac{1}{z+R}-\frac{1}{|z-R|}\right)\right] \\
&= \begin{cases} \dfrac{\sigma R\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left[(z+R)-(R-z)-(z^2-R^2)\left(\dfrac{1}{z+R}-\dfrac{1}{R-z}\right)\right] & \text{if } z<R \\ \dfrac{\sigma R\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left[(z+R)-(z-R)-(z^2-R^2)\left(\dfrac{1}{z+R}-\dfrac{1}{z-R}\right)\right] & \text{if } z>R \end{cases} \\
&= \begin{cases} \dfrac{\sigma R\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left[2z-(z^2-R^2)\left(\dfrac{-2z}{R^2-z^2}\right)\right] & \text{if } z<R \\ \dfrac{\sigma R\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left[2R-(z^2-R^2)\left(\dfrac{-2R}{z^2-R^2}\right)\right] & \text{if } z>R \end{cases} \\
&= \begin{cases} \dfrac{\sigma R\hat{\mathbf{z}}}{4\epsilon_0 z^2}(0) & \text{if } z<R \\ \dfrac{\sigma R\hat{\mathbf{z}}}{4\epsilon_0 z^2}(4R) & \text{if } z>R \end{cases} \\
&= \begin{cases} \mathbf{0} & \text{if } z<R \\ \dfrac{\sigma R^2\hat{\mathbf{z}}}{\epsilon_0 z^2} & \text{if } z>R \end{cases} \\
&= \begin{cases} \mathbf{0} & \text{if } z<R \\ \left(\dfrac{q}{4\pi R^2}\right)\dfrac{R^2\hat{\mathbf{z}}}{\epsilon_0 z^2} & \text{if } z>R \end{cases}
\end{aligned}$$

Therefore, the electric field at $\mathbf{r} = \langle 0,0,z\rangle$ is

$$\mathbf{E} = \begin{cases} \mathbf{0} & \text{if } z<R \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{z^2}\hat{\mathbf{z}} & \text{if } z>R \end{cases}.$$